1. **Problem 2.10**

Particle in a Divided Box

A box containing a particle is divided into a right and a left compartment by a thin partition. If the particle is known to be on the right (left) side with certainty, the state is represented by the position eigenket $|R\rangle$ $(|L\rangle)$*, where we have neglected spatial variations within each half of the box. The most general state vector can then be written as*

$$|\alpha\rangle = |R\rangle \langle R|\alpha\rangle + |L\rangle \langle L|\alpha\rangle \,,$$

where $\langle R|\alpha\rangle$ *and* $\langle L|\alpha\rangle$ *can be regarded as "wave functions." The particle can tunnel through the partition; this tunneling effect is characterized by the Hamiltonian*

$$H = \Delta(|L\rangle \langle R| + |R\rangle \langle L|),$$

where Δ *is a real number with the dimension of energy.*

(a) Find the normalized energy eigenkets. What are the corresponding energy eigenvalues?

(b) In the Schödinger picture the base kets $|R\rangle$ and $|L\rangle$ are fixed, and the state vector moves with time. Supposed the system is represented by $|\alpha\rangle$ as given above at $t = 0$. Find the state vector $|\alpha, t_0 = 0; t\rangle$ for $t > 0$ by applying the appropriate time-evolution operator to $|\alpha\rangle$.

(c) Suppose that at $t = 0$ the particle is on the right side with certainty. What is the probability for observing the particle on the left side as a function of time?

(d) Write down the coupled Schrödinger equations for the wave functions $\langle R|\alpha, t_0 = 0; t\rangle$ and $\langle L|\alpha, t_0 = 0; t\rangle$. Show that the solutions to the coupled Schrödinger equations are just what you expect from (b).

(e) Suppose the printer made an error and wrote H as

$$H = \Delta\, |L\rangle \langle R| \,.$$

By explicitly solving the most general time-evolution problem with this Hamiltonian, show that probability conservation is violated.

Solution.
My goodness look at the size of this problem. These problem types are usually the bane of students struggling to keep up with all of their work, but being able to correctly do these problem types is a great sign for the mastery of whatever the core concept of the problem is. Enough gawking, lets just jump right in.

a.) Starting here, recall that in order to find the energy eigenkets or eigenvalues of a system, you simply utilize the characteristic equation from linear algebra to solve for the eigenvalues, and then subsequently find the eigenkets(eigenvectors). As a review, I'll work this out step by step.

$$H = \Delta(|L\rangle\langle R| + |R\rangle\langle L|) = \begin{pmatrix} 0 & \Delta \\ \Delta & 0 \end{pmatrix}$$

The above comes from remembering that the matrix elements of an operator X can be represented through

$$\begin{aligned} X &= \sum_{a''}\sum_{a'} |a''\rangle\langle a''|X|a'\rangle\langle a'| \\ \to H &= \begin{pmatrix} \langle R|H|R\rangle & \langle R|H|L\rangle \\ \langle L|H|R\rangle & \langle L|H|L\rangle \end{pmatrix}, \end{aligned}$$

(page 20 of Sakurai covers this more in depth if confused). Next, we simply solve the characteristic equation for this matrix,

$$\begin{aligned} |H - \lambda I| &= 0 \\ \to \begin{vmatrix} -\lambda & \Delta \\ \Delta & -\lambda \end{vmatrix} &= 0 \\ \to E_{\pm} &= \pm\Delta \end{aligned}$$

Here, I switched the notation from λ to E to highlight the fact that these are energy eigenvalues, and we will use these to write our resulting eigenkets as well, as that will cleanly highlight which eigenvalue will result from the Hamiltonian acting on that eigenket.

To solve the respective eigenkets we simply plug in the respective eigenvalues, which gives us

$$\begin{aligned} E_+ &: \\ \begin{pmatrix} -\Delta & \Delta \\ \Delta & -\Delta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} &= 0 \to x = y, \\ |E_+\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ E_- &: \\ \begin{pmatrix} \Delta & \Delta \\ \Delta & \Delta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} &= 0 \to x = -y, \\ |E_-\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{aligned}$$

so we now see our eigenvalues and normalized eigenkets can be represented as

$$|E_{\pm}\rangle = |\pm\Delta\rangle = \frac{1}{\sqrt{2}}(|R\rangle \pm |L\rangle)$$

b.) For this portion, we want to find the time evolved state. We know that we can find the time evolved state by applying the time evolution operator to the stationary state,

$$|\alpha(t)\rangle = \hat{U}(t)\,|\alpha(t=0)\rangle$$

where our time evolution operator is of the form

$$\hat{U}(t) = e^{-iHt/\hbar}$$

where H is the Hamiltonian. We thus need to know how the time-evolution operator acts on the state of the system, so we must first expand our stationary state in the energy basis through the insertion of unity,

$$|\alpha\rangle = \sum_n |n\rangle\,\langle n|\alpha\rangle$$

where $|n\rangle$ is an energy eigenstate. Next, we plug in our two energy eigenkets that we previously solved for, and we arrive at

$$|\alpha\rangle = |E_+\rangle\,\langle E_+|\alpha\rangle + |E_-\rangle\,\langle E_-|\alpha\rangle$$

Expanding the $|\alpha\rangle$ we then have

$$|\alpha\rangle = |E_+\rangle\,\langle E_+|\,(|R\rangle\,\langle R|\alpha\rangle + |L\rangle\,\langle L|\alpha\rangle) + |E_-\rangle\,\langle E_-|\,(|R\rangle\,\langle R|\alpha\rangle + |L\rangle\,\langle L|\alpha\rangle)$$

We can then expand the $|R\rangle$ and $|L\rangle$ in terms of our energy eigenkets,

$$|R\rangle = \frac{1}{\sqrt{2}}(|E_+\rangle + |E_-\rangle)$$
$$|L\rangle = -\frac{1}{\sqrt{2}}(|E_-\rangle - |E_+\rangle)$$

while leaving $\langle R|\alpha\rangle$, $\langle L|\alpha\rangle$ untouched as they are simply the "wavefunctions" in either the R or L basis. In doing so, our wavestate expanded in the energy eigenstate basis is

$$|\alpha\rangle = \frac{1}{\sqrt{2}}[(|E_+\rangle + |E_-\rangle)\,\langle R|\alpha\rangle + (|E_+\rangle - |E_-\rangle)\,\langle L|\alpha\rangle],$$

and applying our time evolution operator to the energy eigenstates is straightforward, resulting in

$$|\alpha(t)\rangle = \frac{1}{\sqrt{2}}[(e^{-i\Delta t/\hbar}\,|E_+\rangle + e^{i\Delta t/\hbar}\,|E_-\rangle)\,\langle R|\alpha\rangle + (e^{-i\Delta t/\hbar}\,|E_+\rangle - e^{i\Delta t/\hbar}\,|E_-\rangle)\,\langle L|\alpha\rangle]$$

Which can be factored to simplify, as well as turning the $\Delta/\hbar = \omega$, but I will leave that to the reader to do.
c.) For this part, we realize it is in the right side with certainty at $t = 0$, so our initial wavestate is then

$$|\alpha, t = 0\rangle = |R\rangle$$

due to $\langle R|\alpha\rangle = 1$, $\langle L|\alpha\rangle = 0$, so if we do the same procedure as above to find our time evolved wavestate, we end up with

$$|\alpha(t)\rangle = \frac{1}{\sqrt{2}}[e^{-i\omega t}|E_+\rangle + e^{i\omega t}|E_-\rangle]$$

Now, to find the probability of it being in the left state at a time t, we just apply the postulate

$$P = |\langle L|\alpha(t)\rangle|^2$$

with $\langle L|$ expanded in the energy basis, and we find that

$$\begin{aligned} P = |(-\frac{1}{\sqrt{2}}(\langle E_-| - \langle E_+|))|(\frac{1}{\sqrt{2}}[e^{-i\omega t}|E_+\rangle + e^{i\omega t}|E_-\rangle])|^2 \\ = |\frac{1}{2}[e^{-i\omega t} - e^{i\omega t}]|^2 \\ = \sin^2 \omega t \end{aligned}$$

which just tells us that at a later time t, our probability to observe the particle on the left side varies with time as a function of $\sin^2 \omega t$ with $\omega = \Delta/\hbar$ as previously defined.
d.) This part asks us to write down the coupled Schrödinger equations for the two wave functions. Remember since the state is formed of these two individual wavefunctions, right now we are only solving the Schrödinger equations of the COUPLED systems. So our first step for this is just to write down the general Schrödinger equation, so we know where to go.

$$i\hbar\frac{d}{dt}|\Psi\rangle = \hat{H}|\Psi\rangle$$

Now we write this individually in terms of the wavefunction in the "right" basis, and the wavefunction in the "left" basis, $\langle R|\alpha, t_0 = 0; t\rangle$ and $\langle L|\alpha, t_0 = 0; t\rangle$.

$$\begin{aligned} i\hbar\frac{d}{dt}\langle R|\alpha, t_0 = 0; t\rangle = \hat{H}\langle R|\alpha, t_0 = 0; t\rangle \\ i\hbar\frac{d}{dt}\langle L|\alpha, t_0 = 0; t\rangle = \hat{H}\langle L|\alpha, t_0 = 0; t\rangle \end{aligned}$$

bring the hamiltonian operator in,

$$\begin{aligned} i\hbar\frac{d}{dt}\langle R|\alpha, t_0 = 0; t\rangle = \langle R|\hat{H}|\alpha, t_0 = 0; t\rangle \\ i\hbar\frac{d}{dt}\langle L|\alpha, t_0 = 0; t\rangle = \langle L|\hat{H}|\alpha, t_0 = 0; t\rangle \end{aligned}$$

Now there is a a lot of algebra here and frankly more than I want to type, but essentially just take the form of the Hamiltonian we already have, as well as $|\alpha, t_0 = 0; t\rangle$ that we also have expressed in terms of the energy eigenkets, set $\Delta/\hbar = \omega$, $\langle R|\alpha(t)\rangle = \psi_R$, $\langle L|\alpha(t)\rangle = \psi_L$, and you will arrive at

$$
\begin{aligned}
i\hbar\dot{\psi}_R &= \Delta\psi_L(t) \\
i\hbar\dot{\psi}_L &= \Delta\psi_R(t) \\
\rightarrow \dot{\psi}_R &= -i\omega\psi_L \\
\rightarrow \dot{\psi}_L &= -i\omega\psi_R
\end{aligned}
$$

with the second derivatives being the following

$$
\begin{aligned}
\ddot{\psi}_R &= -\omega^2\psi_R \\
\ddot{\psi}_L &= -\omega^2\psi_L
\end{aligned}
$$

which leads to solutions of the form

$$\psi_R(t) = Ae^{i\omega t} + Be^{-i\omega t},\ \psi_L(t) = Ce^{i\omega t} + De^{-i\omega t}$$

which you can see is the same form as our result for part b, simply set the constants equal to the parameters in b, which are defined as our initial conditions.
e.) This part just notice that the Hamiltonian is not Hermitean, and it results in no non-zero eigenvalues. You can show this doesn't work by realizing generically the time evolution operator in Sakurai is represented as

$$\hat{U}(t)e^{-iHt/\hbar} = 1 - \frac{iHt}{\hbar}$$

and then simply show that

$$\langle\psi(t)|\psi(t)\rangle = \langle\psi(0)|\hat{U}^\dagger\hat{U}|\psi(0)\rangle$$

is time-dependent and cannot be normalized. ■