1. **Problem 1.26**

Transformation Matrix

Construct the transformation matrix that connects the S_z diagonal basis to the S_x diagonal basis. Show that your result is consistent with the general relation

$$U = \sum_r |b^{(r)}\rangle \langle a^{(r)}|$$

Solution.
This problem wants to make sure we know how a transformation matrix or operator works in transforming one space to another, and how we can construct these both generically and specifically. This is important to us in our investigation of the Dirac approach to quantum mechanics, as in Dirac notation, all of how we act on states is defined by operators. For example, the time evolution of a generic wave function from Griffiths is defined as the solution to the partial differential equation of Schrödingers equation, typically in the form of $exp(-iE_n t/\hbar)$, while in Dirac notation we simply define this as the time evolution operator, so if we want to get a state to evolve in time, we simply apply the operator to it.

To solve this, we simply realize that we are transforming from the S_z basis to the S_x basis, so we write

$$|S_x, \pm\rangle = \hat{U} |S_z, \pm\rangle$$

where all we are saying is that $|S_x, +\rangle$ results from a transformation matrix applied to $|S_z, +\rangle$. Next, we see we can obtain the matrix elements of $\hat{U}$ if we simply multiply both sides by $\langle S_z, \pm|$.

$$\langle S_z, \pm|S_x, \pm\rangle = \langle S_x, \pm|\hat{U}|S_z, \pm\rangle$$

Now we just need to know what the $S_x\pm$ basis kets are represented by in terms of the $S_z\pm$ basis kets. We find this as

$$|S_x, \pm\rangle = \frac{1}{\sqrt{2}}(|+\rangle_z \pm |-\rangle_z)$$

where the subscript z is present simply to draw attention to these being the S_z basis kets. Now we simply solve the inner products to find our matrix elements,

$$\begin{aligned}
\langle S_x,+|\hat{U}|S_z,+\rangle &= (\frac{1}{\sqrt{2}}(\langle+| + \langle-|))(|+\rangle) \\
&= \frac{1}{\sqrt{2}} \\
\langle S_x,-|\hat{U}|S_z,-\rangle &= (\frac{1}{\sqrt{2}}(\langle+| - \langle-|))(|-\rangle) \\
&= -\frac{1}{\sqrt{2}} \\
\langle S_x,+|\hat{U}|S_z,-\rangle &= (\frac{1}{\sqrt{2}}(\langle+| + \langle-|))(|-\rangle) \\
&= \frac{1}{\sqrt{2}} \\
\langle S_x,-|\hat{U}|S_z,+\rangle &= (\frac{1}{\sqrt{2}}(\langle+| - \langle-|))(|+\rangle) \\
&= \frac{1}{\sqrt{2}}
\end{aligned}$$

and now that we have all of the matrix elements, we can represent $\hat{U}$ in matrix form as

$$\hat{U} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Just as a quick text, lets see if this transformation matrix transforms the vector form of $|S_z,+\rangle$ to the vector form of $|S_x,+\rangle$ correctly,

$$\begin{aligned}
\hat{U}\,|S_z,+\rangle &\stackrel{?}{=} |S_x,+\rangle \\
\rightarrow \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} \frac{1}{\sqrt{2}}(1) + \frac{1}{\sqrt{2}}(0) \\ \frac{1}{\sqrt{2}}(1) - \frac{1}{\sqrt{2}}(0) \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}
\end{aligned}$$

which in braket notation is

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|+\rangle_z + |-\rangle_z)$$

so our transformation matrix checks out and works(at least for this case, but we will assume the other case as well).

Next, lets just test and see if the formula they gave us works and gives us the same answer.

$$\hat{U} = \sum_r |b^{(r)}\rangle\,\langle a^{(r)}|$$

$$\rightarrow \hat{U} = |S_x,+\rangle\,\langle S_z,+| + |S_x,-\rangle\,\langle S_z,-|$$

Lets just expand the $|S_x, \pm\rangle$ kets in terms of the S_z basis to simplify the outer products we will take, resulting in

$$\hat{U} = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)\langle +| + \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)\langle -|$$

And expanding this we just have

$$\hat{U} = \frac{1}{\sqrt{2}}(|+\rangle\langle +| + |-\rangle\langle +| + |+\rangle\langle -| - |-\rangle\langle -|)$$

$$|+\rangle = \begin{pmatrix}1\\0\end{pmatrix}$$

$$|-\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

from this we can clearly see that this results in

$$\hat{U} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}$$

which perfectly matches our previous result. ■