1. **Problem 1.23**

Compatible and Degenerate Operators

Consider a three-dimensional ket space. If a certain set of orthonormal kets–say, $|1\rangle$, $|2\rangle$, and $|3\rangle$–are used as the base kets, the operators A and B are represented by

$$A \doteq \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix}, \quad B \doteq \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix}$$

with a and b both real.
a.) Obviously A exhibits a degenerate spectrum. Does B also exhibit a degenerate spectrum?
b.) Show that A and B commute.
c.) Find a new set of orthonormal kets that are simultaneous eigenkets of both A and B. Specify the eigenvalues of A and B for each of the three eigenkets. Does your specification of eigenvalues completely characterize each eigenket?

Solution.
a.) We begin this solution by recognizing that in the matrix B, we have a isolated b in the top left spot, which gives should obviously show that one of our eigenvalues is b. Next, without doing any calculations, we should be able to guess that their will be degeneracy, based on the fact that in our remaining submatrix, the absolute value of the offdiagonal components are the same. But, to do the actual calculation, we utilize this block diagonal submatrix, and find the eigenvalues of it using the characteristic equation.

$$\begin{vmatrix} -\lambda & -ib \\ ib & -\lambda \end{vmatrix} = 0$$
$$\lambda^2 = b^2 \to \lambda = \pm b$$

Which shows us that yes, it exhibits degeneracy, because we already have an eigenvalue of b, so now our full spectrum is $\lambda = \pm b$, with b having multiplicity 2.

b.) To solve this, we need to remember what commuting means. Commuting, in the very basic sense, describes how "alike" two quantities are. If two operators commute, that means that they share common eigenstates with each other. Also, remember that observables are described by operators, so lets now jump and say we have two observables that commute, what does that mean? That means that if I do a measurement on my system with A(referring to this problem), the system will jump into an eigenstate of A giving us eigenvalue a. Remember that if we did a measurement immediately after this of A again, we would again get the measurement result (eigenvalue) a. This is because the system collapsed to that eigenstate of A, and the subsequent application of the operator A on that eigenstate is guaranteed to give us the eigenvalue a. Now that that has been reviewed, lets assume that immediately after that measurement of A with result a we do a measurement of B, what value would we get? If A and B commute, we know what we will get, we will get the eigenvalue of B corresponding to that eigenstate, lets call it b. This is because they share eigenstates. I may have meandered some getting to the punchline, but it is essentially this: When two operators (observables) commute, that means we can be certain of the results of subsequent measurements of those two observables, and we can simultaneously measure these observables without destroying the state of the system. After that long roundabout talk about the importance of commutation, how do we actually solve it? Well, we define the commutator of two operators as such

$$[A, B] = AB - BA$$

And the two operators are said to commute if

$$[A, B] = 0$$

And remember order matters here because they are operators, so we then have

$$[A, B] = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix} \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix} - \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix}$$

Do a bit of linear algebra, and we see that it is true for this system, $[A, B] = 0$.
c.) If you are confused here, it will help to remember that kets are essentially described by column vectors. Having said that, we see that $|1\rangle$ is a common eigenket of both A and B, with eigenvalues a and b respectively. Also, A is diagonalized, so it is easy to describe all the corresponding eigenstates and eigenvalues of A as

$$a\,|1\rangle\,,\ -a\,|2\rangle\,,\ -a\,|3\rangle$$

where we see $|2\rangle\,, |3\rangle$ have degenerate eigenvalues(-a). We will write this first orthonormal ket that is a simultaneous eigenket of both A and B in the form

$$|a, b\rangle = |1\rangle$$

Next, we remember that we can reduce B to block diagonal form and just try to find the corresponding eigenvectors of that submatrix of B. First, let us find the corresponding eigenvector of B when the eigenvalue is +b.

$$\begin{pmatrix} -b & -ib \\ ib & -b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which after reducing gives us an eigenvector corresponding to the eigenvalues of -a and +b of

$$|-a, b\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

We do this same procedure but with an eigenvalue of -b, which then gives us the result of

$$|-a, -b\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Which means we have simultaneous eigenstates of both operators described by

$$|a, b\rangle \,, \; |-a, b\rangle \,, \; |-a, -b\rangle$$

Before we end this problem, lets realize the importance of what we just did. Lets say given a system where A and B are both observables that we want to do measurements of, if we can write our state in terms of these simultaneous eigenstates, we can automatically find the probability of each measurements result, and more without doing anything complicated. ■