1. **Example 2.5**

SHO Expectation value of potential energy

Lets investigate example 2.5 from Griffiths, which should be on page 47 of your text if you have the third edition. We will not be directly following the example as we will be implementing a Dirac approach similar to that of section 2.3 of Sakurai. This example essentially asks the student to find the expectation value of the potential energy in the case of the Simple Harmonic Oscillator for any n state.

$$\langle V \rangle = ?,\ V = \frac{1}{2}m\omega^2 x^2$$

Solution.
In approaching any type of SHO problem we should immediately seek to apply the creation and annihilation operators, as they beautifully operate on the states of the system, and we have previously derived their relation to other operators for this potential, allowing us to represent nearly all of our operators in terms of only these two, providing a simple approach to these problem types. Lets begin by writing our position operator in terms of these two operators (we don't need momentum or the hamiltonian),

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)$$

those following from Griffiths may be concerned with the notation of $a, a^\dagger$, but I urge you not to worry, as these are simply the a_-, a_+ you are used to seeing, respectively. Also, you may have noticed me referring to them as the "creation/annihilation" operators instead of the "raising/lowering" verbage that Griffiths utilizes. I prefer calling them the former simply due to the idea of the operator "creating" or "destroying" an excited state seems more intuitive to me.

Now, getting back on track. We know we need to find how x^2 acts on this state, not just x, so we square the position operator that was in the form we just stated, arriving at

$$\hat{x}^2 = \frac{\hbar}{2m\omega}(a^2 + {a^\dagger}^2 + a^\dagger a + a a^\dagger)$$

Special care must be made when approaching this problem to remember that $a, a^\dagger$ are operators, and the order of multiplication matters here($a^\dagger a \neq a a^\dagger$). Now we will simply bring in the multiplicative constant that effects x^2 to turn it into V and simplify, landing us at

$$\hat{V} = \frac{\hbar\omega}{4}\hat{x}^2$$

Now we should realize the only operator component of V is $\hat{x}^2$, meaning we only need to find out the expectation value of x^2 and then multiply by a constant to get our final answer. Now, lets switch to dirac notation and simply say that

$$\langle x^2 \rangle = \langle n|x^2|n\rangle = \langle n|a^2 + {a^\dagger}^2 + a^\dagger a + aa^\dagger|n\rangle$$

Now, I jump and say that

$$\langle x^2 \rangle = \langle n|a^\dagger a + aa^\dagger|n\rangle$$

If you don't get this jump, work through it. Essentially we know that for the end inner product all of the non n eigenstates will result in non contributions to the expectation value, so we remove the first two terms due to them only raising or lowering the n eigenstate.

Actually doing the calculation we end up with

$$\langle x^2 \rangle = (n + (n+1))\langle n|n\rangle = 2(n + \frac{1}{2})$$

Now we plug this back in to get our potential energy, and find that the expectation value for the potential energy for any n state is given as

$$\langle V \rangle = \frac{\hbar\omega}{2}(n + \frac{1}{2})$$

Which means that the potential energy for any n state is exactly half that of the total energy, with the other half being covered by kinetic energy. This naturally makes sense, but does it make sense classically? Surprisingly, in this case (opposing from what we usually expect) this quantum result does actually match what we expect classically. You may say "But wait! The SHO classically has great sinusoidal oscillations, meaning there are points when kinetic is max and potential is min, not the same!", and that is true, but remember we are looking at the expectation value here. The expectation value is essentially the mean of many measurements, and if we took many measurements on a SHO that had no dissipative forces, we would end up seeing potential and kinetic are both half of the total. ■