1. **Problem 11.2.1**

Holomorphic Real Function

Show whether or not the function $f(z) = \mathrm{R}(z) = x$ is analytic.

Solution.
We begin this solution by realizing that a test we use to see whether or not a complex function is holomorphic is the *Cauchy-Riemann* equation test. Now, in order to use the CR equations to test for this, we must first recognize that we can write our complex functions in the form of

$$f(z) = f(x + iy) = u(x, y) + iv(x, y)$$

This comes from our generic definition of complex space being R^2, and from recognizing that we can split our complex function into both a real and imaginary component. Rewriting our function in this form, we see that

$$\begin{aligned} f(z) &= x + i(0) \\ u(x, y) &= x \\ v(x, y) &= 0 \end{aligned}$$

Now, we can use the CR equations as a first test for analyticity, which to show the CR equations are

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}$$

with for our specific function,

$$\begin{aligned} u_x &= 1 \\ u_y &= 0 \\ v_x &= 0 \\ v_y &= 0 \end{aligned}$$

So it can be seen it does not satisfy the CR equations, therefore is not analytic. ■

2. **Problem 11.2.11**

Two-dimensional irrotational fluid flow

Two-dimensional irrotational fluid flow is conveniently described by a complex potential $f(z) = u(x, v) + iv(x, y)$. We label the real part, $u(x, y)$, the velocity potential, and the imaginary part, $v(x, y)$, the stream function. The fluid velocity V is given by $\mathbf{V} = \nabla u$. If $f(z)$ is analytic:
(a) Show that $df/dz = V_x - iV_y$
(b) Show that $\nabla * \mathbf{V} = 0$ (no sources or sinks).
(c) Show that $\nabla \times \mathbf{V} = 0$ (irrotational, nonturbulent flow).

Solution.
a.) We can solve this by realizing once again that $z = x + iy$ for a complex function, and that for z to change slightly (a derivative) then x or y must change slightly (i is i, it cant change).

$$df/dz = du/dx + idv/dx + (du/dy + idv/dy)$$
$$V_x = du/dx$$
$$V_y = du/dy$$
$$\rightarrow df/dz = V_x + V_y + i(dv/dx + dv/dy)$$

We can see that we are close to an answer, but not quite. Notice our V_y does not have an imaginary component, and we have all these secondary derivatives of v. Lets think, either our approach is wrong, or we can simplify this further. Lets try simplifying further before we tackle another approach. We can simplify the derivatives of v by utilizing the CR equations from the previous problem,

$$u_x = v_y$$
$$u_y = -v_x$$

So for our case we then have

$$v_x = -u_y$$
$$v_y = u_x$$

Plugging in to our equation we then have

$$df/dz = V_x + V_y + i(-V_y + V_x)$$

We now see we are much closer to our desired answer, but we have a few too many factors here. We realize we can get to our desired answer by saying that instead of $df/dz = df/dx + df/dy$, we simply say $df/dz = df/dx$, which gives us

$$df/dz = df/dx = V_x - iV_y$$

which means our initial assumption of $df/dz = df/dx + df/dy$ was wrong, and in fact we can just say $df/dz = df/dx$.
b.) To prove this we plug in our fluid velocity $\mathbf{V} = \nabla u$, which gives us

$$\nabla * (\nabla u) = 0$$

From this we recognize that the divergence of the gradient of a function is called the Laplacian, ∇^2, so we can rewrite this as

$$\begin{aligned} \nabla^2 u &= 0 \\ u_{xx} + u_{yy} &= 0 \end{aligned}$$

Again, we return to the CR equations to finish solving this.

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x \\ \to (v_y)_x + (-v_x)_y &= 0 \\ \to u_{yy} &= -u_{xx} \\ \to \nabla^2 u &= 0 \end{aligned}$$

c.) Again, we begin by substituting our fluid velocity expression in,

$$\nabla \times (\nabla u) = 0$$

which is the curl of the gradient of u, which we can write as

$$\begin{aligned} \begin{vmatrix} \delta/\delta x & \delta/\delta y \\ \delta u/\delta x & \delta u/\delta y \end{vmatrix} &= \delta/\delta x(\delta u/\delta y) - \delta/\delta y(\delta u/\delta x) \\ &= \frac{\delta^2 u}{\delta x \delta y} - \frac{\delta^2 u}{\delta y \delta x} \end{aligned}$$

Also, we know from general calculus that

$$\frac{\delta^2 u}{\delta x \delta y} = \frac{\delta^2 u}{\delta y \delta x}$$

so naturally we arrive at

$$\nabla \times (\nabla u) = \frac{\delta^2 u}{\delta x \delta y} - \frac{\delta^2 u}{\delta y \delta x} = 0$$

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3. **11.4.2**

Cauchy's Integral Formula

Evaluate

$$\oint_C \frac{dz}{z^2 - 1}$$

Where C is the circle $|z - 1| = 1$.

Solution.
We approach this problem through the use of the Cauchy integral formula, which states that:

$$\frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz = f(z_0)$$

We can already see our integral is very close to this form, and the bottom can be factored as

$$z^2 - 1 = (z - 1)(z + 1)$$

which leads to our integral being the following (I label it as I to follow the convention of the book),

$$I = \oint_C \frac{1}{(z - 1)(z + 1)} dz$$

which, without taking heed of C, gives us two singularity points at $z = \pm 1$. If we take into account that C is a circle of radius 1 centered at 1, then within our boundary we only have a single singularity point, that being $z = 1$ which is a singularity on our boundary.

$$f(z) = \frac{1}{z + 1}$$
$$\rightarrow \frac{1}{2\pi i} \oint \frac{f(z)}{z - 1} dz = f(z_0)$$
$$\rightarrow \oint \frac{f(z)}{z - 1} dz = 2\pi i f(z_0)$$
$$f(z_0) = \frac{1}{2}$$

So we then have as our solution for this integral,

$$\oint_C \frac{dz}{z^2 - 1} = \pi i$$

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